ISyE/Math/CS/Stat 525 – Linear Optimization
Spring 2020

Assignment 3

Due date: March 6 at 11:59pm.

Instructions and policy: This assignment contains two sections, one including mandatory exercises, and one with optional exercises, that serve for extra-practice.

Undergraduate students should handle in only the mandatory exercises that are marked with [U].
Graduate students should handle in only the mandatory exercises that are marked with [G].

The assignments should be submitted electronically in Canvas. Late submission policy: 10% of total points will be deducted per hour. IMPORTANT: Plan on submitting well before the deadline. If a technical problem occurs, and you cannot resolve it by the deadline, send an email to the TA before the deadline and attach your solution.

Students are strongly encouraged to work in groups of two on homework assignments. To find a partner you can post on the "Discussions" section in Canvas. Only one file should be submitted for both group members. In order to submit the assignment for your group please follow these steps in Canvas: Step 1. Click on the "People" tab, then on "Assignment", and join one of the available groups for the assignment; Step 2. When also your partner has joined the same group, one of the two can submit the assignment by clicking on the "Assignments" tab, then on the assignment to be submitted, and finally on "Submit assignment". The submission will count for everyone in your group.

Groups must work independently of each other, may not share answers with each other, and solutions must not be copied from the internet or other sources. If improper collaboration is detected, *all groups* involved will automatically receive a 0. Students must properly give credit to any outside resources they use (such as books, papers, etc.). In doing these exercises, you must justify all of your answers and cite every result that you use. You are not allowed to share any content of this assignment.

Compulsory exercises

Exercise 1 [U][G] .... ***5 points***

Consider the polyhedron P defined by the following system of 5 linear inequalities in 3 variables x_1, x_2, x_3.

$$\begin{array}{llll} 2x_1 & -5x_2 & +4x_3 & \leq 10 \\ 3x_1 & -6x_2 & +3x_3 & \leq 9 \\ 5x_1 & +10x_2 & -x_3 & \leq 15 \\ -x_1 & +5x_2 & -2x_3 & \leq -7 \\ -3x_1 & +2x_2 & +6x_3 & \leq 12 \end{array}$$

Apply the Fourier-Motzkin elimination algorithm to P to compute $\Pi_1(P)$ by eliminating first variable x_3 and then variable x_2. Is P empty?

Exercise 2 [U] .... ***5 points***

Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$, and assume that the rows of the matrix A are linearly independent.

(a) (2 points) Suppose that two different bases lead to the same basic solution. Show that the basic solution is degenerate.

(b) (2 points) Consider a degenerate basic solution. Is it true that it corresponds to two or more distinct bases? Prove it or give a a counterexample.

(c) (1 point) Suppose that a basic solution is degenerate. Is it true that there exists a distinct adjacent basic solution which is degenerate? Prove it or give a a counterexample.

Exercise 3 [U][G] .... ***6 points***

Let $A_1, \dots, A_n$ be a collection of vectors in $\mathbb{R}^m$.

(a) (3 points) Let
$$C = \left\{ \sum_{i=1}^{n} \lambda_i A_i : \lambda_1, \dots, \lambda_n \geq 0 \right\}.$$
Show that if $y \in C$, then there exist coefficients $\lambda_1, \dots, \lambda_n \geq 0$ such that (i)*at most* m of the coefficients are nonzero, and (ii) $y = \sum_{i=1}^{n} \lambda_i A_i$.

Hint: Consider the polyhedron
$$Q = \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i A_i = y, \lambda_1, \dots, \lambda_n \geq 0 \right\}.$$

(b) (3 points) Let P be the convex hull of the vectors A_i, i.e.
$$P = \left\{ \sum_{i=1}^{n} \lambda_i A_i : \sum_{i=1}^{n} \lambda_i = 1, \lambda_1, \dots, \lambda_n \geq 0 \right\}.$$
Show that if $y \in P$, then there exist coefficients $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, such that (i) *at most* $m+1$ of the coefficients are nonzero, and (ii) $y = \sum_{i=1}^{n} \lambda_i A_i$.

Exercise 4 [G] .... ***5 points***

Consider the polyhedron $P = \{x \in \mathbb{R}^n : a_i' x \geq b_i, i = 1, \dots, m\}$. Suppose that u and v are *distinct* basic feasible solutions that satisfy $a_i' u = a_i' v = b_i, i = 1, \dots, n-1$, and assume that the vectors $a_1, \dots, a_{n-1}$ are linearly independent (this implies that u and v are adjacent basic feasible solutions). Let $L = \{\lambda u + (1-\lambda) v : 0 \leq \lambda \leq 1\}$ be the segment that joins u and v. Prove that $L = \{z \in P : a_i' z = b_i, i = 1, \dots, n-1\}$. (*Hint: Consider the one-dimensional set* $G = \{z \in \mathbb{R}^n : a_i' z = b_i, i = 1, \dots, n-1\}$.)

Exercise 5 [U][G] .... ***6 points***

Let $P = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1, x \geq 0\}$ and consider the vector $x = (0, 0, 1)$. Find the set of feasible directions at x.

Exercise 6 [U][G] .... ***8 points***

Consider the problem of minimizing $c'x$ over a polyhedron P. Prove the following:

(a) (6 points) A feasible solution x is optimal if and only if $c'd \geq 0$ for every feasible direction d at x.

(b) (2 points) A feasible solution x is the unique optimal solution if and only if $c'd > 0$ for every nonzero feasible direction d at x.

Optional exercises

Exercise 7 .... ***0 points***

We know that every linear program can be transformed into an equivalent linear program in standard form. We also know that a nonempty polyhedron in standard form has at least one extreme point. We are then tempted to conclude that every nonempty polyhedron has at leat one extreme point. What is wrong with this argument?

Exercise 8 .... ***0 points***

Recall that a set $S \subset \mathbb{R}^n$ is said to be *convex* if for any $x, y \in S$, and any $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in S$.

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function and let $S \subset \mathbb{R}^n$ be a convex set. Let x^* be an element of S. Suppose that x^* is a local optimum for the problem of minimizing $f(x)$ over S; that is, there exists some $\varepsilon > 0$ such that $f(x^*) \leq f(x)$ for all $x \in S$ for which $\|x - x^*\| \leq \varepsilon$. Prove that x^* is a global minimum; that is, $f(x^*) \leq f(x)$ for all $x \in S$.

Exercise 9 .... ***0 points***

Consider the set $\{x \in \mathbb{R}^n \mid x_1 = x_2 = \cdots = x_{n-1} = 0,\ 0 \leq x_n \leq 1\}$. Could this be the feasible set of a problem in standard form in $\mathbb{R}^n$?